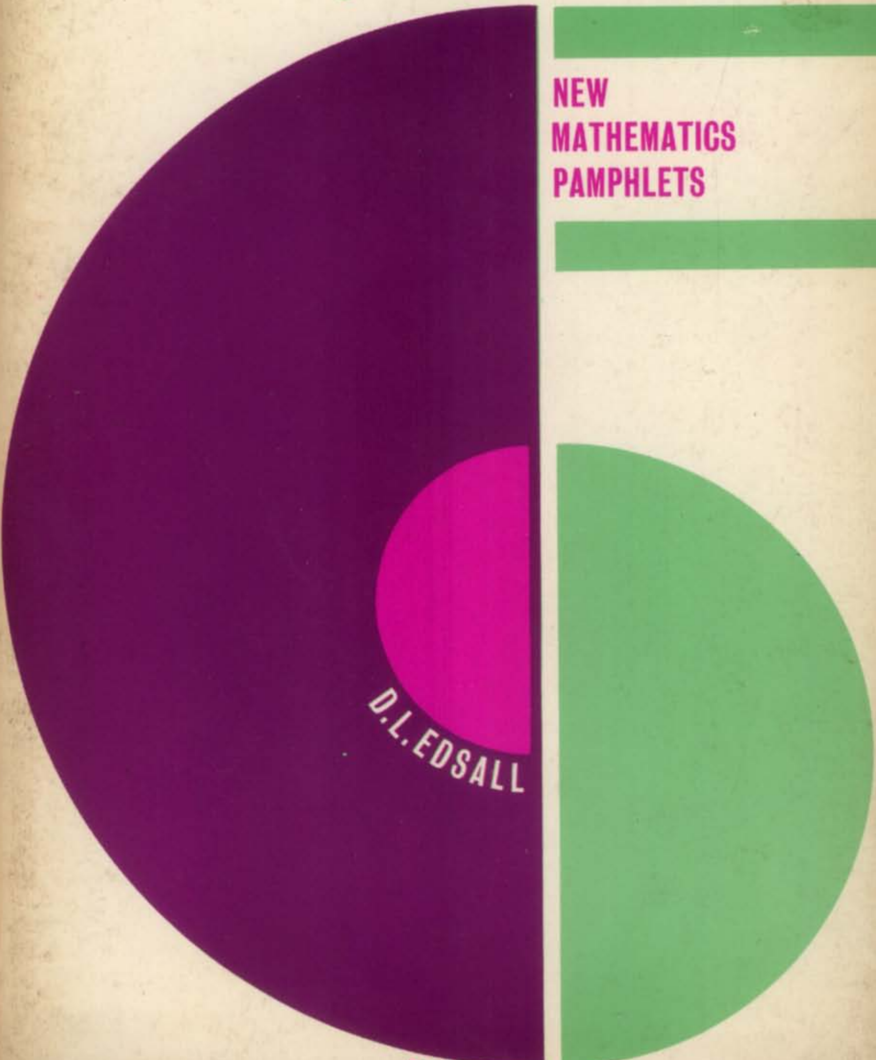
GRAPHS AND INEQUALITIES



NEW
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D.L. EDSALL

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NEW MATHEMATICS PAMPHLETS

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SETS

VECTORS AND MATRICES IN TWO DIMENSIONS

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Introduction

A large amount of the subject matter of this pamphlet has been taught to junior forms for a long time, but I feel that by including the newer ideas of inequalities the pupil will obtain a more thorough understanding of the uses of graphical representations and of coordinates. As these are the basic methods that the mathematician uses to describe the positions and movements of the objects that he wants to study, I feel that they should form an important part of any junior mathematics course. I have started the pamphlet from the very basic beginnings of coordinates, and although it is suitable for a pupil who has done some graphical work before starting this course, no particular previous knowledge is assumed.

I would like to thank Mr. R. M. Dunnett for his assistance in checking the exercises and in the reading of the proofs of the text.

GRAPHS AND INEQUALITIES

In this pamphlet we shall be interested in the relative sizes of numbers, and simple expressions which involve numbers. We shall investigate some of the properties by using accurate drawings on graph paper, and at the same time consider what algebraic processes have been achieved as a result of our drawings. We will start with an easy problem.

Take the following set of numbers, and rewrite them in ascending order of magnitude (that is, with the smallest first and steadily increasing).

$$7,\ -5,\ 1,\ 0{\cdot}9,\ -3,\ -1{\cdot}25,\ 0,\ 4{\cdot}25,\ 2,\ 1{\cdot}1,\ -1{\cdot}4$$

You certainly should not have any difficulty with the positive numbers in this set but notice that -3 is less than -2 (written in symbols as $-3 < -2$). If you find this puzzling, wait until we illustrate our numbers in a diagram, and then see how this result fits in. In order to get this diagram, draw a line, and select a point on it to represent 0. (As there are numbers from -5 up to $+7$ this point should be somewhere near the middle of the line.) Choose a scale along the line, say each unit is represented by half an inch. Then start marking off half-inch sections along the line, starting at the point you have selected as 0, and marking these points successively 1, 2, 3, 4, etc. If you start on the other side of 0 and do the same thing, you get -1,

-2, -3, -4, etc. Thus a number is represented by a point at a distance from 0 proportional to its magnitude; the side of 0 on which it appears is determined by its sign. Thus 1·1 will be one and one-tenth units from 0 on the side which we have agreed to call *positive*. It is conventional to consider the side "to the right" of 0 as the positive side. The numbers above are illustrated in figure 1.

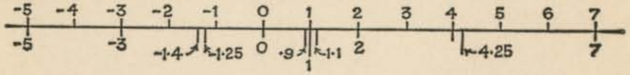

FIG. 1

Now if we want all the numbers x which are greater than 3 ($x > 3$) we shall be interested in all those points that are further away from 0 than 3, i.e. all the points along the line which are "to the right" of 3. Similarly if we want all numbers greater than -3, then we consider all numbers represented by points that are "to the right" of -3. In particular we see that $-2 > -3$. Moreover it should be clear that if we have two numbers, both of which are greater than 2, one of which (x) is much greater, the other (y) is only just greater, then the gap between 2 and x will be larger than the gap between 2 and y, i.e. the greater the difference between x and 2, the further away is x from 2.

To summarize what we have obtained so far: if we consider some variable number x, then if $x = 5$ we can illustrate this by a point 5 units away from a fixed point along a certain line. If $x > 5$, we are considering all the points along the line that are further from the fixed point than 5 (and on the same side of it). Such a collection of points (or numbers) is called a *set* of points (or a *set* of numbers). The term *set* implies that there is a clearly defined rule so that we can tell whether any point (or number) is a member of the set or not. "The set of all numbers x such that x is greater than 5" may be written

symbolically as $\{x \mid x > 5\}$. Notice that 5 itself is not a member of this set.

The point $x = 5$ is also connected with another set $\{x \mid x < 5\}$ and this is the set of all points "to the left" of 5. Again 5 is not a member of this set, and both these sets are called *open sets*, for we can go as close as we like to 5, but the actual boundary point representing 5 is not included in the set. If we include the boundary as part of the set by saying, for example, "x is to be greater than or equal to 5" (written as $x \geqq 5$) then we call the set *closed*. Examples of open sets are $\{x \mid x > 2\}$, $\{x \mid 4 < x < 5\}$, $\{x \mid 4 < x \leqq 5\}$, $\{x \mid x < 7\}$ and the corresponding closed sets are $\{x \mid x \geqq 2\}$. $\{x \mid 4 \leqq x \leqq 5\}$, $\{x \mid 4 \leqq x \leqq 5\}$ and $\{x \mid x \leqq 7\}$.

EXERCISE 1

(Any question may be done by using a diagram.)

1. Draw a line as in figure 1 and mark in with different colours the sets of points represented by:

(i) $\{x \mid x < -1\}$; (ii) $\{x \mid x > 2\}$; (iii) $\{x \mid -2 < x < 2\}$; (iv) $\{x \mid -3 \leqq x < 2\}$; (v) $\{x \mid -1 \leqq x \leqq 1\}$.

2. Which of the sets in question 1 are closed sets?

3. If x is a number such that $x \geqq -1$ and also $x \leqq 2$, write down the set to which x belongs, using the same notation as in question 1.

4. If $-1 \leqq x < 2$ and also $-2 < x < 1$, to what set does x belong? Is this an open or a closed set?

5. Are there any values of x which satisfy the following pairs of conditions? If so write them down.

(i) $-1 < x < 2, x > 1$; (ii) $-1 < x \leqq 2, x > 2$; (iii) $-1 \leqq x \leqq 2, x \geqq 2$.

6. If x obeys either one or the other or both of the following pairs of conditions, write down all the values which x can take (using the notation of question 1) and illustrate your answer with a coloured diagram.

(i) $x > 2, x > 3$; (ii) $x > 2, x < 3$; (iii) $-1 < x < 2, x < -1$; (iv) $-3 < x < 2, -2 \leqq x \leqq 2$; (v) $x > -1, x < 3$.

7. I have a car which has a maximum speed of 90 m.p.h. The place that I want to go to is along a road which has a 40-m.p.h. speed limit. On actually making the journey I never go slower than 10 m.p.h. If x stands for the average speed for the journey, write down three inequalities which x must obey. If the distance travelled is 10 miles, what are the greatest and least times that the journey would take?

8. A pendulum is swinging backwards and forwards about a point O. The farthest it swings is to 10° to either side of O. If $x°$ represents the angle between it and O at a given instant, what inequality must x obey?

The following questions are suitable for those readers who have done some set theory.

9. Write down the intersections of the following sets:

(i) $\{x \mid x > 2\}, \{x \mid -1 \leqq x \leqq 3\}$;
(ii) $\{x \mid 1 \leqq x < 2\}, \{x \mid x \leqq 1\}$;
(iii) $\{x \mid 1 < x < 3\}, \{x \mid x \leqq -1\}$.

10. Write down the unions of the following pairs of sets:

(i) $\{x \mid -1 < x < 2\}, \{x \mid x > 1\}$;
(ii) $\{x \mid -1 < x < 2\}, \{x \mid x > 2\}$;
(iii) $\{x \mid x > -1\}, \{x \mid x > -2\}$;
(iv) $\{x \mid -1 \leqq x \leqq 1\}, \{x \mid -1 < x < 2\}$.

11. In which of the pairs of sets in question 10 is one set a subset of the other?

12. Finding a union of two sets as in question 10 is equivalent to which of the following: (i) all the numbers x which obey the two conditions at the same time or (ii) all the numbers x which obey either one condition or the other or both?

So far we have considered only the simplest kind of inequality and shown how it can be illustrated in a diagram. We want now to develop the algebra which allows us to investigate inequalities of the form $3x+5 > 7$ or $2-6x < -5$ and which can change them into the simple inequalities of the last section. We shall therefore consider what happens to our diagram when the basic processes of arithmetic are performed on the numbers represented in it. For simplicity we shall use the set $\{-3, -1, 0, 1, 2, 4\}$ to illustrate what is happening, but the results will be true for any set of numbers. Firstly we will add 2 to each number and get the set $\{-1, 1, 2, 3, 4, 6\}$.

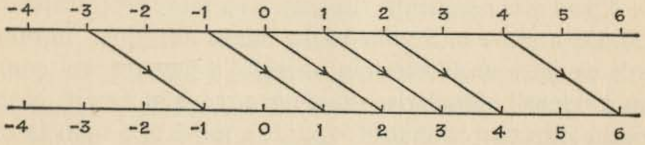

FIG. 2

By looking at the diagram (figure 2) it is clear that all that has happened is that each number has been displaced 2 units along the line "to the right"—note that their relative positions have remained unchanged. This means that if we have a number x such that $x-2 > 1$ we may add 2 to each side of the inequality getting $(x-2)+2 > 1+2 \Rightarrow x > 3$. (The sign $\Rightarrow$ means "implies that".) All that has happened to the diagram is a displacement by 2 units in the positive direction.

Similarly a subtraction of 2 will give the set

$$\{-5,\ -3,\ -2,\ -1,\ 0,\ 2\}$$

which is just the original set displaced two units "to the left" (figure 3). Again this will not affect the order of the numbers.

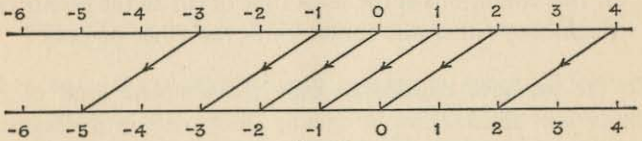

FIG. 3

This means that if we have $x+2 < 5$, then we may subtract 2 from each side and so get $(x+2)-2 < 5-2 \Rightarrow x < 3$. Notice how subtraction is the reverse of addition—subtracting 2 is the thing which "undoes" the previous addition of 2. Notice also how we must treat each side of the inequality in the same way in order that we may be sure of getting a true result.

The direction in which the move is made is determined by the sign. $-(+5)$ is a move of 5 units in the opposite direction to $+5$, and is consequently the same as a move represented by (-5), i.e. a move of 5 units in the minus direction. In other words we have an illustration of why "a plus times a minus gives a minus". Similarly $-(-5)$ is a move of 5 units in the opposite direction to that of -5, i.e. a move of 5 units in the positive direction, and so is equivalent to $+5$.

The other main process of arithmetic is multiplication. If each number in the original set is multiplied by 2 it becomes $\{-6,\ -2,\ 0,\ 2,\ 4,\ 8\}$. We see that there has been an "expansion" so that the points are now twice as far apart as they were before (figure 4). Again it is clear that the order in which the points appear has not changed, and so any inequalities involved have not changed either. For example

$$\tfrac{1}{2}x > 3 \Rightarrow \tfrac{1}{2}x\,.\,2 > 3\,.\,2 \Rightarrow x > 6$$

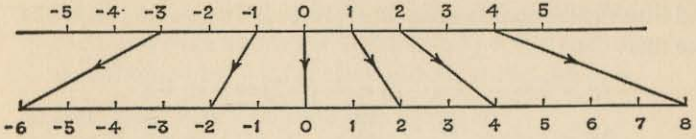

FIG. 4

Division has a similar effect because dividing by 2 is the same as multiplying by $\frac{1}{2}$; the difference is that the diagram has been contracted instead of expanded (figure 5), and the set becomes

$$\{-\tfrac{3}{2},\ -\tfrac{1}{2},\ 0,\ \tfrac{1}{2},\ 1,\ 2\}$$

Thus $$2x < 9 \Rightarrow 2x\,.\,\tfrac{1}{2} < 9\,.\,\tfrac{1}{2} \Rightarrow x < 4\tfrac{1}{2}.$$

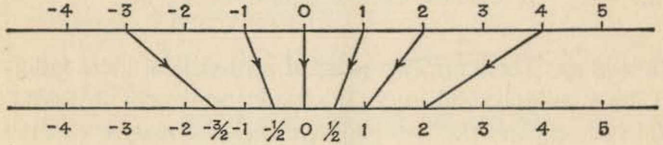

FIG. 5

Multiplication by a negative number is a somewhat more complicated process for it involves two different effects. It has been shown that -2 is the same thing as $-(+2)$ and it is in this way we should think of multiplication by -2. The set of numbers becomes $\{6,\ 2,\ 0,\ -2,\ -4,\ -8\}$ and not only has there been an expansion of 2 times but also the order in which the numbers should be considered has been reversed (figure 6); for, if they are considered in ascending order, they become

$$\{-8,\ -4,\ -2,\ 0,\ 2,\ 6\}$$

It is the minus sign which has caused this reversal of the order, and thus the inequalities become reversed. In algebra this looks like this:

$$-\tfrac{1}{2}x > 4 \Rightarrow -\tfrac{1}{2}x.(-2) < 4.(-2) \Rightarrow x < -8$$

or $$-2x < -6 \Rightarrow -2x.(-\tfrac{1}{2}) > -6.(-\tfrac{1}{2}) \Rightarrow x > 3$$

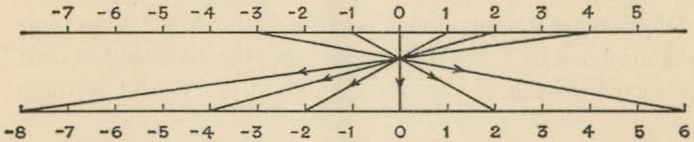

FIG. 6

Question: All the lines in figure 6 pass through a certain point. Why? Is this also true for figures 4 and 5?

To sum up: There are two pairs of arithmetical ideas [addition and subtraction] and [multiplication and division]. Within these pairs the two processes are the reverse of each other—addition by 2 "undoes" subtraction by 2 and vice versa; multiplication by 2 "undoes" division by 2 and vice versa. Providing that we use these processes to operate on each side of the inequality in the same way, we shall not affect it unless we multiply or divide by a negative number; then the minus sign will cause the inequality to be reversed.

EXERCISE 2

(Any of the following questions may be done with the aid of an accurate diagram.)

1. Draw diagrams to illustrate the set of numbers $\{-2, -1, 1, 3, 5\}$ and what happens to each number when (i) 3 is added to each one, (ii) 4 is subtracted from each, (iii) each number is multiplied by 3, (iv) each number is multiplied by -3, (v) each number is divided by 4.

2. Find what values of x are permissible if
(i) $x+3 > 7$, (ii) $x+2 < 3$, (iii) $x+5 > -1$, (iv) $x-5 < -1$, (v) $x-3 > 2$, (vi) $\dfrac{x}{3} < 1$, (vii) $\dfrac{x}{5} < -2$, (viii) $-\dfrac{x}{7} > 2$, (ix) $2x < 5$, (x) $2x > -3$, (xi) $-3x < 7$, (xii) $-5x > -6$.

3. Find what values of x are permissible if
(i) $3(x+7) > 6$, (ii) $5(x-2) < 10$, (iii) $7(x+3) < -14$, (iv) $2(6-x) > 12$, (v) $-3(6+x) > 12$, (vi) $\tfrac{1}{3}(2-x) > -1$, (vii) $-\tfrac{1}{8}(4-2x) < -5$.

4. If $3(x+5) \leqq 21$ and $2(7-x) \leqq 10$, can you say anything about the possible values for x?

5. (i) If $a > b$ and $b > c$, does it automatically follow that $a > c$?
(ii) If $a > b$ and $b < c$, can you say anything about the relative sizes of a and c?

The following questions are of a problem nature, and you will not be able to solve them using only the ideas which are in the text.

6. Is $1{\cdot}414 > \sqrt{2}$?

7. (i) If $x^2 < 9$, does this imply that $x < 3$?
(ii) If $x^3 < 27$, does this imply that $x < 3$?
(iii) Why is there a difference in these two cases?

8. If $(x-1)(x-2) > 0$, what values may x take? (*Hint.*—If two numbers are multiplied together and the answer is positive, what can you say about the possible signs for the two numbers?)

9. If $(x+2)(x+1) < 0$, what values may x take?

10. If $x > px$, what can you say about the values of p (i) when $x > 0$, (ii) when $x < 0$?

All the inequalities so far have involved only one unknown number, usually represented by x. We want to go on to consider expressions involving two unknowns, for which we shall use the letters x and y. The kind of expression to be considered is $3x-2y$, and in order to consider these expressions we need a way of illustrating them. For this we need graph paper. Previously one line was necessary to illustrate the numbers, but now that there are two unknown numbers we shall need two lines drawn on the graph paper. These are called *axes*—the x-axis and the y-axis, and for convenience they will be drawn at right angles. They meet at a point O which is called the *origin*, and this point will represent 0 both for x and y. Just as before, we set up a scale along each line (these scales need not necessarily be the same) and plot values of x along the x-axis and values of y along the y-axis. The effect is shown in figure 7. The arrows indicate the direction of increase for x and y. These two lines split the paper into four parts called *quadrants*. For identification purposes these are numbered as in figure 7.

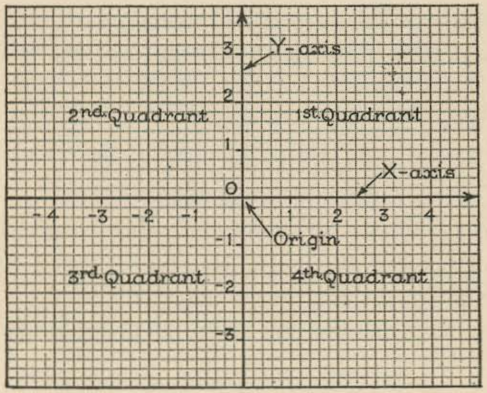

FIG. 7

In order to illustrate a pair of values, say $x = 2$, $y = 3$, we go along the x-axis 2 units and then move along a line parallel to the y-axis through this value (MP in figure 8); x will have the value 2 at any point of this line. We then consider the y-value of 3, and move along a line parallel to the x-axis through this point (NP in figure 8) until we meet the first line. This point P will have the two required values $x = 2$, $y = 3$. These two values are called the *coordinates* of P. The x-coordinate is always written first, the y-coordinate always second, and they are written in brackets thus: $(2, 3)$. It is important to realize that the x-coordinate must always come first, for the pair $(3, 2)$, i.e. $x = 3$, $y = 2$, gives a completely different point Q (figure 8 illustrates this). The points $(2, 3)$ and $(3, 2)$ are both in the first quadrant; the point $(-3, 1)$ will be in the second quadrant, for its x-coordinate is negative, whilst its y-coordinate remains positive. Similarly $(1, -2)$ is in the fourth quadrant (see figure 8).

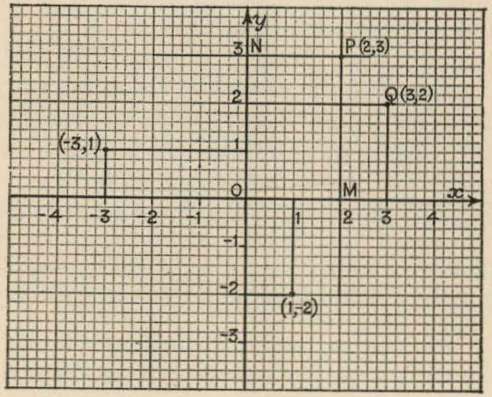

FIG. 8

Notice specially that all the points where x has the value 2 lie along the line PM produced in both directions. $x = 2$ is said to be "the equation of the line PM". $x = 3$ would be a similar line parallel to the y-axis, and three units from it. $y = 3$ is the equation of PN produced in both directions.

We now go back to the expression $3x-2y$, and work out its values for various values of x and y. These will be given as coordinates, i.e. for the pair $(1,3)$ we have $x = 1$, $y = 3$, and so $3x-2y = 3.1-2.3 = 3-6 = -3$. Take each pair of values below and work out the value of $3x-2y$. Make up a table showing the value you have worked out against the pair of values of x and y for which it was calculated.

$(-3,-3)$ $(0,0)$ $(-2,-3)$ $(6,6)$ $(0,-3)$ $(-1,2)$ $(-2,2)$ $(1,5)$
$(-3,-1)$ $(2,-2)$ $(-1,0)$ $(3,3)$ $(1,3)$ $(4,3)$ $(4,6)$ $(3,6)$.

Now take a piece of graph paper, draw two axes on it such that with a scale of $\frac{1}{2}$ inch for a unit for both x and y, all the points whose coordinates appear above can be plotted on the paper. Mark out the values along the axes (as in figure 8) and then plot each of the above points with a neat cross. Beside this cross put the corresponding value of $3x-2y$, which you have just worked out. When you have done this with all sixteen points, look at the pattern of the values you have put beside each of the points. There should be several important properties that can be spotted from these numbers. Make a list of the ones that you can see before reading further.

The points that should have been noted are as follows:

1. If you join all the points which have the same value they should lie along straight lines, and these straight lines are all parallel to each other.

2. The line which joins all the points with 0 as value (i.e. where $3x-2y = 0$) splits the paper into two parts such that all the values on one side are positive (i.e. $3x-2y > 0$) and on the other all the values are negative (i.e. $3x-2y < 0$). Moreover the further away from the 0-value line, the greater the positive values become, and the less the negative ones.

3. If you move from one point which you have plotted where the value is 0 to the next along the line, you always move two units in the x-direction and three units in the y-direction.

4. The properties given in 2 and 3 are also true for any fixed value of $3x-2y$. Thus there is a line along which $3x-2y$ is always equal to -3 and this divides the plane into two parts representing $3x-2y > -3$ and $3x-2y < -3$. Also moving from one point to the next along the line will mean a move of 2 units in the x-direction and 3 units in the y-direction.

5. As you move along a line where all the values of x are the same (say $x = 2$) then the values of $3x-2y$ go down by 2 as y goes up 1 unit in value. Similarly if you move along a line where y always has the same value (say $y = -1$) $3x-2y$ changes value by 3 as x increases by 1 unit.

The most important of these properties are the first four. The first three points are developments of the work which we have already done with one unknown number. The comparison of these ideas is as follows:

1. $x = -3$ was represented by a point on a line, and this point divides the line into two parts $x > -3$ and $x < -3$. Now $3x-2y = -3$ is a line which divides the plane of the graph paper into two parts $3x-2y > -3$ and $3x-2y < -3$.

2. The further we move away from the point $x = -3$ the greater the difference between x and -3 will be. Now the further we move away from the line $3x-2y = -3$ the greater the difference between $3x-2y$ and -3 becomes; all the points which are the same distance away will lie along a line parallel

to $3x-2y = -3$ and all give the same value to the inequality, i.e. we get lines parallel to the original one which represent all the points such that $3x-2y = 0$, $3x-2y = 6$, etc.

Now suppose that x and y are two numbers such that $3x-2y < -3$ and also such that $4x+y > 0$. The question is, are there any restrictions on the values which either x or y may take? We shall consider a solution using an accurately drawn graph.

The first step is to find which area of the paper represents $3x-2y < -3$. To do this we have first to draw the line $3x-2y = -3$ and then discover which side of the line represents the inequality that we want.

1. To draw the line $3x-2y = -3$ we select any three values of x (say $x = -1, 0$, and 1) and then work out the corresponding values of y which are $y = 0$, $\frac{3}{2}$, and 3 respectively. The points $(-1, 0)$ and $(0, \frac{3}{2})$ and $(1, 3)$ are then plotted. Check that they do lie in a straight line (if not there is a mistake somewhere in the working) and draw it in with a ruler.

2. To find which side of this line represents $3x-2y < -3$ we need only investigate what happens at one point. The origin $(0, 0)$ is usually the most convenient for this purpose. Substituting we get $3x-2y = 0-0 = 0$ which is not < -3. Consequently all points on the origin side of the line are on the side which we do NOT want. To show this we shade out that side of the line as in figure 9.

The process is now repeated with $4x+y > 0$. Plotting points for the line $4x+y = 0$ we have $(-1, 4)$, $(0, 0)$, $(1, -4)$. Since the origin now lies on the line itself, it is not a suitable point to use to find which side represents the inequality that we want. Consequently we use another point instead—say $(1, 0)$. This gives $4x+y = 4 > 0$ and so is on the correct side of the line.

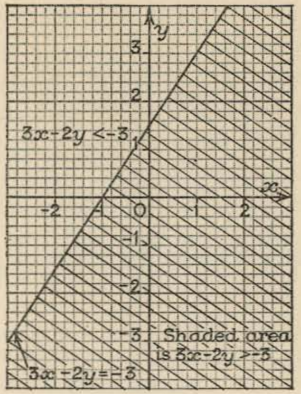

FIG. 9

Finally shading out the area in which we are not interested we get figure 10. In this figure only the area which has not been shaded at all is permissible. Clearly in order to get into this area y must be greater than the value where the lines cut (approximately 1·09). The line represented by say $y = -1$ will

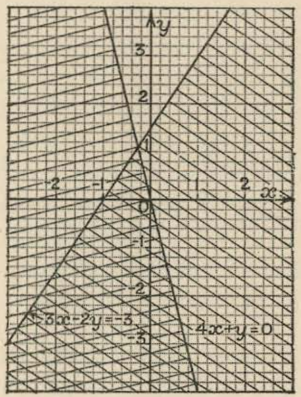

FIG. 10

not cut the non-shaded area at all, and so no value of x can be found. On the other hand as the non-shaded area is V-shaped, if we were to increase it indefinitely upwards it would spread out to allow any value of x to be used, providing that y is already $> 1{\cdot}09$. Thus we must have $y > 1{\cdot}09$, but subject to this condition x may take any value. If we want to see how this kind of problem could be tackled algebraically we have:

(i) To find the restrictions on the values of x and y implied by $x-y<2$ and $x+y>4$.

$$\begin{aligned} &x-y<2 \Rightarrow x<2+y \Rightarrow -x>-2-y \\ &x+y>4 \Rightarrow y>4-x \end{aligned}$$

and putting these together

$$y>4-x>4-2-y \Rightarrow y>2-y \Rightarrow 2y>2 \Rightarrow y>1$$

(ii) To find the restrictions on the values of x and y implied by $4x+y>0$ and $3x-2y<-3$. (This was the question that was solved graphically in the last paragraph.)

$$\begin{aligned} &4x+y>0 \Rightarrow y>-4x \\ &3x-2y<-3 \Rightarrow 3x<2y-3 \Rightarrow x<\tfrac{2}{3}y-1 \Rightarrow -x>1-\tfrac{2}{3}y \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \Rightarrow -4x>4-\tfrac{8}{3}y \end{aligned}$$

Therefore

$$y>-4x>4-\tfrac{8}{3}y \Rightarrow y+\tfrac{8}{3}y>4 \Rightarrow \tfrac{11}{3}y>4 \Rightarrow y>\tfrac{12}{11}$$

But if we try to do the same for x we get:

$$\begin{aligned} &4x+y>0 \Rightarrow 4x>-y \Rightarrow x>-\tfrac{1}{4}y \\ &3x-2y<-3 \Rightarrow -2y<-3x-3 \Rightarrow -\tfrac{1}{4}y<-\tfrac{3}{8}x-\tfrac{3}{8} \end{aligned}$$

and these two inequalities cannot be fitted together for a final deduction. Consequently there is no restriction to the values of x.

EXERCISE 3

1. Draw and shade a diagram to illustrate the area represented by $2y-x>2$; on the same piece of graph paper shade the area represented by $y+6x>12$. If x and y are such that $2y-x>2$ and at the same time $y+6x>12$, what can you say about possible values for x and y?

2. If in question 1 we add a further condition that $x+y<6$, what whole-number values of x and y will satisfy all three inequalities?

3. If $y-2x<6$ and $2y-x>4$, what are the restrictions on the values that x and y may take?

4. If in question 3 we have a third condition that $x+y<2$, what whole-number values of x and y satisfy all three conditions?

5. If $2y+5x<-4$ and $y<4x$, find any restrictions that there are to possible values of x and y.

6. In a certain examination there is a wide choice of questions from two sections of the paper. Questions in section A score 5 marks each, questions in section B score 20 marks each, and a candidate must score 80 marks to pass. Questions from section A take, on average, 10 minutes each, and those from section B 15 minutes each. The candidate is allowed 2 hours for the paper. What are the possible combinations of questions from each section which would give the required pass mark? Illustrate your answer with a diagram.

7. Write down the comparisons (1) and (2) of the two-variable case with the one-variable case (see page 19) using the language of set theory.

8. Use the language of set theory to describe the non-shaded area in figure 10.

9. Explain how the set identity $A' \cap B' \equiv (A \cup B)'$ is illustrated by figure 10.

10. What would you expect to be the difference between an open and a closed set in two dimensions (i.e. involving 2 unknowns x and y)?

So far we have not used the fact that the further we go away from a line such as $3x-2y=5$, the greater becomes the difference in value from 5—either $3x-2y>5$ or $3x-2y<5$ as the case may be. This property comes in useful for solving many practical problems, and methods similar to the following one are constantly in use to solve production problems in industry and commerce. To show how this idea is applied we shall solve a problem.

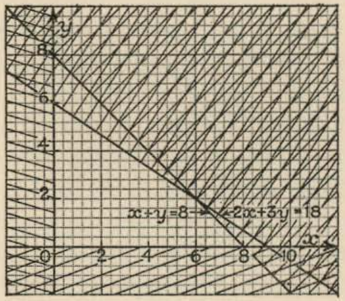

FIG. 11

A boy goes off to buy sweets at the tuckshop. He has 1/6d to spend, and he decides to buy wine gums and some toffee. There is a school rule which says that he may not buy more than 8 oz of sweets at a time. The prices are: wine gums 2d per oz and toffee 3d per oz. He reckons that one ounce of wine gums will give him 15 minutes chewing time and one ounce of toffee 20 minutes. How many ounces of each should he buy in order to get the maximum chewing time for the money that he spends?

Let him buy x oz of wine gums and y oz of toffee. Then clearly x and y must be positive numbers so we have (i) $x>0$, (ii) $y>0$. From the school rule we must have that (iii) $x+y \leqq 8$.

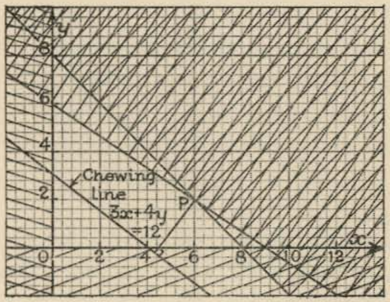

FIG. 12

Finally since he has only 1/6d to spend we have (iv) $2x+3y \leqq 18$. Drawing the appropriate lines and shading out unwanted area for the inequalities (i) to (iv) we get a diagram as in figure 11.

Now the chewing time is $\frac{1}{4}x+\frac{1}{3}y$ hours, and we want to consider a series of lines representing different chewing times to find which will be the greatest. If we consider a chewing time of 1 hour we have a "chewing line" of $\frac{1}{4}x+\frac{1}{3}y=1$, i.e. $3x+4y=12$. If we draw this in we see that quite a length of it lies inside the "permitted area" (figure 12). Thus there are various possible combinations of amounts of these two kinds of sweets that could give him a chewing time of 1 hour. Moreover the origin $(0, 0)$ gives a point such that $\frac{1}{4}x+\frac{1}{3}y < 1$, so all the chewing times of less than 1 hour lie on the origin side of the line $3x+4y = 12$. Consequently all the chewing times greater than 1 hour lie on that side of the line opposite to the origin, and the further we move away from $3x+4y = 12$ the greater the chewing time becomes. But we are still restricted to the "permitted area" (unshaded) and closer inspection shows that the point furthest away from $3x+4y = 12$ that lies in this area is P. Consequently the values of x and y at P will give the greatest chewing time permissible under the conditions that have been laid down. Therefore the greatest chewing time that the boy can buy corresponds to $x = 6$ and $y = 2$ (i.e. 6 oz of wine gums and 2 oz of toffee) and he will then get a chewing time of $(\frac{1}{4}.6+\frac{1}{3}.2) = \frac{3}{2}+\frac{2}{3} = \frac{13}{6} = 2$ hours 10 minutes.

EXERCISE 4

1. If $x > 0$, $y > 0$, $5x+6y < 30$, $7x+4y < 28$, what is the greatest value of $x+y$?

2. If $0 < x < 4$, $0 < y < 5$, $6x+5y < 30$, what is the greatest value of $2y+5x$?

3. A smallholder owns a field of area 12,000 square yards which he wants to plant with apple trees. He has available £1000 to spend on the trees. Two types of tree do well in his area. Type A requires 9 square yards of ground per tree; it produces on average 30 lb of fruit per tree, which sells at a profit of 3d per lb; and costs £1 per plant.

Type B requires 16 square yards of ground per tree; produces an average of 50 lb of fruit per tree, which sells at a profit of 2d per lb; and costs 25/– per plant.

How many of each kind of tree should be planted for the maximum profit?

4. A bus company is about to order a new fleet of buses for a certain area of operation. It can choose from:

Type A—a double-decker which requires a crew of 2, which will carry 60 passengers and cost £4800.

Type B—a single-decker which requires a crew of 1, which carries 40 passengers and costs £3600.

The company is prepared to pay up to £48,000 for the new buses, and it has available a maximum number of 15 trained crew. If, on average the profit on type A is 2/– per day per passenger seat and on type B is 2/6d per day per passenger seat, what would be the most profitable combination for the company to buy?

5. A boy is training for the football season. He has the choice of training on the field (running, passing, kicking, etc.) or circuit training in the gym. He reckons that he can spare 5 hours for training each week, and would prefer to spend more time on the field than in the gym. He reckons that (i) he must do at least one hour per week on the field to improve his game technique, (ii) apart from that, the gym training is twice as effective as the field training for getting him physically fit for the game. How long should he spend on each type of training if he is to reach a maximum state of fitness for the game? (You should treat these two different kinds of training as independent.)

6. The following problems either have no solution, or many equally valid solutions. Explain why.

(i) If $x > 0, y > 0, x+y < 9, 2x+y < 0$, find the maximum value of $6x+3y$.

(ii) If $x > 0, y > x, x+y < 9, 0 < y < 4$, find the maximum value of $2x+3y$.

(iii) If $x > 0,\ y > 0,\ x+y > 1,\ x+y < 4,\ x > y$, find the maximum positive value of $y-3x$.

Another important property which we observed on p. 19 was that if we moved along one of the lines joining points of equal value (say $3x-2y = -3$) we moved two units in the x-direction as compared with three units in the y-direction. The actual length of the move does not matter—it is always 2 parts x-wards as compared with 3 parts y-wards. This can be stated another way—if we make any move along the line resulting in some change in x, then there must be a corresponding change in y that is $\frac{3}{2}$ times as large (figure 13).

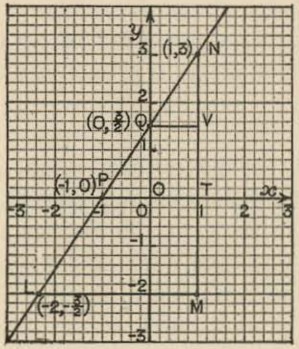

FIG. 13

To illustrate this further we will consider the line $y = 4x-7$.

A basic table of values to be used for this line has been drawn up (figure 14).

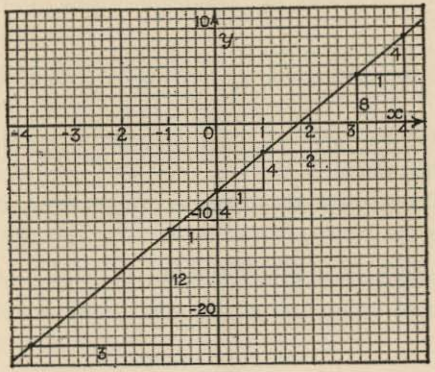

FIG. 14

Values of x	-4		-1		0		1		3		4		7		12
Corresponding values of y	-23		-11		-7		-3		5		9		21		41
Changes in values of x		3		1		1		2		1		3		5	
Corresponding changes in values of y		12		4		4		8		4		12		20	
$\dfrac{\text{Corresponding changes in } y}{\text{Change in } x}$		4		4		4		4		4		4		4	

We find that the rate of change in y is 4 times that in x. These comparative rates of change ($\frac{3}{2}$ in the first case, and 4 in the second) are called the *gradients* of the lines. The steeper the line, the faster y changes as compared with x, and the larger the value of the gradient. (You will find illustrations of this fact in the examples of the next exercise.) Notice that the value of the gradient of the line is obtained by considering "a change in x", then the corresponding change in y when we move along the line, and then forming the fraction

$$\frac{\text{corresponding change in } y}{\text{change in } x}$$

The actual value of the change in x in the first place does not matter.

The gradient is a very important idea in the practical applications of mathematics. It measures comparative rates of change of two things and appears in several disguises. For example, if x measures the time taken to travel a certain distance and y the distance, then the gradient will be

$$\frac{\text{a certain distance travelled}}{\text{the time taken}}$$

which is the average speed at which the journey is made (or the actual speed if that is constant). Alternatively if x cwt. is a certain weight of fertilizer, and y is the amount of money paid for it, the gradient is

$$\frac{\text{total price paid}}{\text{amount bought}}$$

and so is the price per hundredweight. Other cases are illustrated in Exercise 5. One thing to stress is that for a straight line, the gradient is the same everywhere along the line (if you have done a little elementary trigonometry, you should recognize that the gradient is the tangent of the angle that the line makes with the x-axis). This property holds only for straight-line graphs; the study of the gradients of curved graphs is rather more complicated, and forms an important branch of mathematics called *calculus*, which you may study later in your school career.

EXERCISE 5

1. Find the gradients of the following lines, by drawing the graph of the line and reading off values of x and y from the graph.
(i) $y-3x = 6$, (ii) $2x-5y = 10$, (iii) $3y-4x = 12$.

2. Find the gradients of the lines which pass through the following pairs of points, using $\dfrac{\text{corresponding change in } y}{\text{change in } x}$

(i) $(1, 2)$ and $(3, 7)$, (ii) $(-5, 6)$ and $(3, 8)$, (iii) $(2, -4)$ and $(-4, -10)$.

Draw the graphs of the three lines on one piece of paper to illustrate the effect of the different values of the gradients.

3. Using the same idea as in question 2, find the gradients of the lines whose equations are:
(i) $3x-5y = 15$, (ii) $2x-4y = 7$, (iii) $3x-7y = 21$.

4. Draw the graphs of the lines which pass through the point $(1, 2)$ and have gradients

(i) 1, (ii) $\frac{3}{4}$, (iii) 0, (iv) 10.

5. Draw the line which passes through the point $(-1, 4)$ and has gradient of 2. Another line is drawn through $(-1, 4)$ and it has a gradient of < 2. Shade an area on your graph paper where this new line must lie.

6. Draw the graph of the line which passes through the points $(1, 5)$ and $(3, 1)$. Using the method of question 2, find its gradient. What do you notice that is different about the gradient of this line as compared with the gradients that you have worked out so far? In what way does the graph of this line vary from the graphs of questions 1, 2, and 3? (Be careful over signs.)

7. Find the gradients of the following lines:

(i) $x+5y = 5$, (ii) $2x+3y = 6$, (iii) $3x-4y = 10$, (iv) $3x+5y = 15$.

8. By comparing the values of the gradients you have worked out in questions 2 and 7 with the equations of the lines themselves, can you see any connection between the numbers involved?

9. The price of sugar is 10d per lb. Draw a graph which may be used to read off the price of any amount up to 4 lb. Where do gradients come into this question?

10. A certain brand of portable oil-fired radiator is designed to heat 100 cubic feet of air through 3 centigrade degrees in half an hour. Draw graphs to show the effect on the temperature of a room of 100 cubic-feet capacity which starts at freezing point (0°C) and into which (i) one, (ii) two, (iii) four such radiators are put. (You may assume that the rate of rise in temperature remains constant.)

11. The radius of a circle is steadily increased at a rate of 1 in per min. At what rate does the circumference increase? Illustrate the fact that this is a constant rate of increase by drawing a graph.

12. Water is leaking at a constant rate from a burst pipe into a cylindrical-shaped bucket. At one instant the depth is 3 in, and five minutes later the depth is 5 in. The bucket is 13 in high, how long will it take to overflow? If we were to draw a graph of depth plotted against time, why would it be a straight line?

13. A man starts from Cambridge at 10 a.m. to walk to Ely (16 miles away) at 4 m.p.h. After $\frac{3}{4}$ hour he meets a bus coming from Ely and going to Cambridge (where it waits 10 min and then returns to Ely). The bus travels at an average speed of 20 m.p.h. Draw a graph to find when the bus will overtake the man. (Taking 1 in = 30 min and 1 in = 2 miles, plot distances vertically.)

14. From Bedford to Stamford is 45 miles. Two cyclists, one from each town, start to ride towards one another. They pass at a point 25 miles from Bedford, and the first cyclist reaches Stamford 1 hr 10 min before the second reaches Bedford. Find their speeds (assumed constant) and the time at which they pass one another.

15. To travel from home to work I can either drive the car at an average speed of 20 m.p.h., or I can go by train at an average speed of 45 m.p.h. If I travel by train, it is necessary to allow 20 minutes at each end for the journeys to and from the stations. What is the shortest distance that it is quicker to go by train, rather than by car, if the distance between the stations may be taken to be equal to the distance between home and work?

16. A line is drawn through the points (3, 4) and (6, 2). Another point on the line is $(8, h)$. Use the idea of gradients to find the value of h.

17. We wish to make up a timetable for trains running from London to King's Lynn via Bishop's Stortford, Audley End, Cambridge, and Ely. Two trains travelling in the same direction may only pass each other at Bishop's Stortford and Cambridge. Draw a diagram to illustrate the journey of the following trains. Assume that their speed from one station to the next is constant, and select suitable figures of your own for these speeds.

9.00 London–King's Lynn, stopping at all stations.
9.20 London–Cambridge–King's Lynn.
9.30 London–Ely, stopping at all stations.
9.50 Coal train non-stop London–Cambridge.
10.10 Fast passenger train non-stop London–King's Lynn.

(The distances are London to Bishop's Stortford 31 miles, Bishop's Stortford to Cambridge 26 miles, Cambridge to King's Lynn 45 miles.)

In the graphs involving travel problems in Exercise 5, we used the point on the graph where lines cut each other to read off some of the answers. This is a very common practice. If there are two lines one with equation $2x+3y=7$, the other with equation $5x-3y=7$, then the point (x, y), such that x and y satisfy the conditions for both lines, must be the point where they meet; for, if it satisfies the conditions, it must lie on both lines, and the point of intersection is the only point to do this. Thus using a graph to find the values of x and y where two lines cut is equivalent to solving the two equations (which represent the lines) simultaneously by the processes of algebra.

EXERCISE 6

1. Solve the equations $3x+5y=8$, $x+4y=5$ algebraically, and illustrate the answer by means of a graph.

2. Solve the following problem (i) algebraically, (ii) by drawing graphs. A square box with a lid is to be made from a piece of card 15 in by 18 in by cutting away the shaded area (see figure 15). Find the values of the length of the base (x) and the height (y).

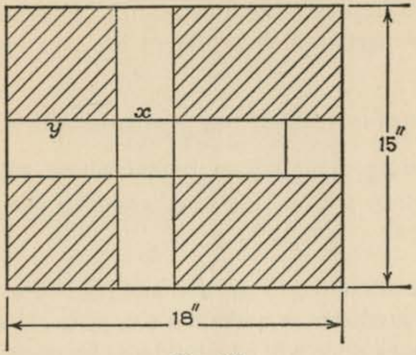

FIG. 15

3. Write down question 1 using the language of set theory.

All the work which has been done so far has been with graphs of expressions which have turned out to be straight lines. Such expressions are called *linear* for this reason. The expression $lx+my+n=0$ will always be linear where x and y are the unknowns being plotted, and l, m, and n are any constant numbers. (If you are not sure what is meant by this, turn back to the equations which appear in Exercise 5 and compare them with the expression $lx+my+n=0$.) These graphs are rather special cases. Almost any expression where y can be expressed in terms of x (y is then said to be a function of x) can be illustrated by plotting the points whose coordinates are (x, y), and in general these graphs will be curves rather than straight lines. Nevertheless these curves possess many of the properties which we have discussed in terms of straight lines so far. The three main properties which we have considered all have their counterpart for curves as well as lines:

(i) The curve (representing an equation) separates the paper into two parts which represent the two inequalities $>$ or $<$.

(ii) The rates of increase of y as compared with x.

(iii) The meeting of curves at points whose values may be calculated by considering simultaneously the equations of the curves.

We shall be interested in ideas (i) and (iii). (The study of the properties of gradients represents a complete subject in itself.) How this may be done will be considered by some examples.

We shall consider these properties with reference to the graph of the expression

$$y = x^2 - x - 2 = (x+1)(x-2)$$

In order to plot the graph of this expression we must first work out the coordinates of points which lie on the curve. We make up a table of values of y for given values of x. We shall use values of x between -4 and 5.

x	-4	-3	-2	-1	0	1	2	3	4	5
$x+1$	-3	-2	-1	0	1	2	3	4	5	6
$x-2$	-6	-5	-4	-3	-2	-1	0	1	2	3
y	18	10	4	0	-2	-2	0	4	10	18

We now choose scales on the axes of x and y so that we can fit these values on to a piece of graph paper (figure 16).

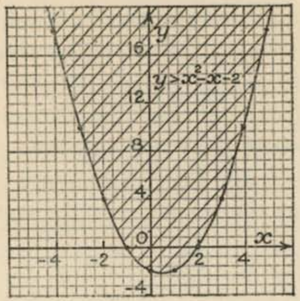

FIG. 16

We then plot the points (x, y) on the graph paper, and the curve can be sketched in. It must be realized that the curve will be smooth, and if there is any difficulty in telling just how the curve should go, then extra points may be worked out and plotted to assist in the drawing. In this case it helps to discover just how the curve "turns round" between $x = 0$ and $x = 1$. Consequently we work out the value of y when $x = \frac{1}{2} \Rightarrow y = (\frac{1}{2}+1)(\frac{1}{2}-2) = (\frac{3}{2})(-\frac{3}{2}) = -\frac{9}{4}$ and plotting $(\frac{1}{2}, -\frac{9}{4})$ we can sketch in the curve.

This curve is called a *parabola* (it occurs as the shape of a reflector of a car headlamp, or approximately the curve in which a stone moves when it is thrown). If we now consider the coordinates of points "inside" the parabola we have

value of x	-2	-1	-1	0	0	1	1	2	2
value of y for a point on the curve	4	0	0	-2	-2	-2	-2	0	0
value of y at a point inside the parabola	6	3	12	0	6	9	0	1	4

From this it will be seen that for all the points that have been worked out $y > x^2-x-2$. This will be true, in fact, for all points in this part of the paper. Similarly all the points "outside" the parabola will obey the inequality $y < x^2-x-2$: Thus the curve divides the paper into two parts. The curve itself is $y = x^2-x-2$, and the two parts are $y > x^2-x-2$ and $y < x^2-x-2$. Note the similarity to the case of the straight-line graph.

Now suppose that we draw the line $y = 2x+3$, then the points where this line and our previous graph meet (figure 17) are points which satisfy simultaneously $y = x^2-x-2$ and $y = 2x+3$, i.e. the x-values are the solutions of $x^2-x-2 = 2x+3 \Rightarrow x^2-3x-5 = 0$. Using this idea of a line intersecting the curve, we can solve various equations from the graph of $y = x^2-x-2$. The following three examples illustrate some possibilities.

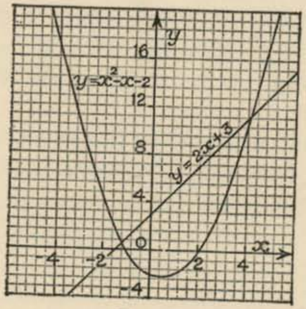

FIG. 17

Example 1

Use the graph of $y = x^2-x-2$ to find values of x that satisfy $x^2-x-7 = 0$.

The first step is to rearrange the equation so that one side of the equals sign becomes x^2-x-2. In this case we get

$$x^2-x-2 = 5 \qquad (a)$$

Now we already have the graph of $y = x^2-x-2$ and if we solve this simultaneously with the equation $y = 5$, we would get equation (a). Consequently if we now draw the graph of $y = 5$, and read off the values of x where the two graphs cut, we have the solutions to equation (a). This is illustrated in figure 18.

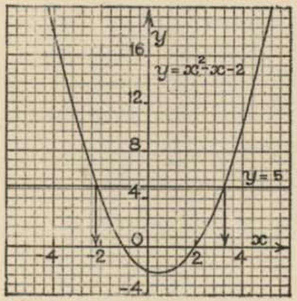

FIG. 18

Example 2

Use the graph of $y = x^2-x-2$ to solve the equation $x^2-2x-1 = 0$.

The first step is the same as in the last example, and rewriting we get

$$x^2-x-2 = x-1 \qquad (b)$$

Now if we consider where our original graph $y = x^2-x-2$ cuts $y = x-1$, then the values of x will be those that satisfy equation (b) and are therefore those required. Consequently we must draw the line $y = x-1$, and then read off the values of x where the two graphs cut. This is illustrated in figure 19.

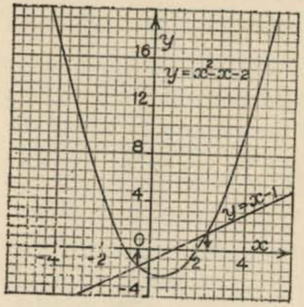

FIG. 19

Example 3

Use the graph of $y = x^2-x-2$ to solve the equation

$$x = \frac{10}{x-1}$$

In this case the equation that we have to tackle would appear very different from x^2-x-2. But if we start by cross-multiplying (or multiplying each side by $x-1$) we get $x^2-x = 10$ and then, proceeding as before, $x^2-x-2 = 8$. So we require the x-values of the intersection of the original graph with the line $y = 8$ (figure 20).

N.B. The degree of accuracy obtained for the answers depends on the scale chosen for the axes of the graph. The larger the scale, the greater the degree of accuracy possible.

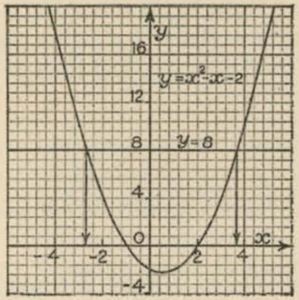

FIG. 20

EXERCISE 7

(Note: scales should be chosen so as to fit the graphs sensibly on to the graph paper in your possession.)

1. Draw a graph and indicate by shading an area that represents $y > x^2$. (Take values of x such that $-4 \leqq x \leqq 4$.)

2. On the same piece of paper as question 1 indicate by shading areas that represent (i) $y > x^2+1$, (ii) $y > x^2+2$. How are these related to $y = x^2$?

3. Draw a graph of $y = x^2+x-5$ for $-4 \leqq x \leqq 4$, and indicate by shading an area that represents $y > x^2+x-5$ and also $y < -1$. (Notice that we may write

$$-1 > y > x^2+x-5 \Rightarrow -1 > x^2+x-5 \Rightarrow 0 > x^2+x-4.)$$

What is the connection between the area that you have shaded and the roots of the equation $x^2+x-4 = 0$?

4. Draw the graph of $y = x^2$ for $-6 \leqq x \leqq 6$. Use this graph to find the solutions to the following equations:

(i) $x^2 = 5$ (ii) $x^2 = x+1$ (iii) $2x^2-x-4 = 0$

(iv) $x = \dfrac{7}{x}$ (v) $x = \dfrac{8}{x+1}$ (vi) $x-2 = \dfrac{8}{x+1}$

5. A boy throws a stone in the air. After x seconds its height is y feet, where $y = 40x - 16x^2$. Plot a graph to illustrate this equation for $0 \leqq x \leqq 3$, and then answer the following questions:

(i) At what time does the stone hit the ground?
(ii) What is the greatest height of the stone?
(iii) For how long is the stone higher than 20 ft?

6. Using just one sheet of graph paper draw the graphs of $y = 7 - x^2$ and $y = x$ for $-4 < x < 4$. Read off the values of x where these graphs cut and find a quadratic equation for which these values are the approximate answers.

7. Draw the graph of $y = 3x^2 + 2x - 1$ for $-3 \leqq x \leqq 3$. By drawing other lines on this graph find the answers to the following questions:

(i) The roots of the equation $3x^2 + 2x - 1 = 4$.
(ii) The roots of the equation $3x^2 + 2x - 10 = 0$.
(iii) The range of values of x such that $3x^2 + 2x - 1 < 10$.
(iv) The least value of $3x^2 + 2x - 1$.
(v) The roots of the equation $3x^2 + 2x - 1 = x$.

8. On the same piece of graph paper draw the graphs of (i) $y = \frac{2}{x}$ and (ii) $y = x^2$ for $1 \leqq x \leqq 4$. Use these graphs to find an approximate value for $\sqrt[3]{2}$. For which part of the diagram is $x^2 > \frac{2}{x}$?

9. Use the graph of $y = x^2 + x + 1$ to solve the following equations:

(i) $x^2 + x - 3 = 0$ (ii) $x^2 - 3x + 1 = 0$

(iii) $x = \dfrac{9}{x+1}$ (iv) $x^2 = 70$

(v) $x + 1 = \dfrac{9}{x+1}$

10. Use graphs to find for what values of x we have $3x + 1 > x^2 - x + 2$.

11. x and y are given in terms of a number t by the equation $x = 3 + t$, $y = 1/t$ where $1 \leqq t \leqq 4$. Using axes for x and y as usual, plot the graph which represents the equation connecting x and y. For what value of t does $x = 3y$?
(t is called a *parameter*, and this form for the equation of the curve is called a *parametric representation*.)

12. Draw the graph of $y = x^3 - x$ for $-3 \leqq x \leqq 3$. For what values of x is $x^3 - x < 0$? By factorizing $x^3 - x$ and considering the signs of each bracket, produce an argument to justify your answer which is not dependent on the graph.
(This curve is a typical graph of a cubic equation. Why must all cubic equations in x always have at least one root? Is this true for (i) quadratic equations, (ii) fourth-degree equations in x?)

13. A farmer wishes to enclose a piece of land. He has available 200 yards of fencing, and he places this in the shape of a rectangle of length x yards and width z yards. Write down (i) an equation connecting x and z, (ii) an expression for the area enclosed in terms of x. By drawing a graph find the greatest possible area he can enclose. What shape of field does this give?

GRAPHS AND INEQUALITIES

The following are some curves which occur frequently in mathematics.

Name of curve	*Normal form of the equation*	*Form of the equation used for plotting*
(1) Circle	$x^2+y^2=9$	$y=\pm\sqrt{(9-x^2)}$

(*a*) Why are there no values for $x>3$? (*b*) What is the value of the radius? (*c*) What geometrical property does the equation $x^2+y^2=9$ represent?

(2) Ellipse	$\frac{x^2}{4}+\frac{y^2}{9}=1$	$y=\pm\frac{1}{2}\sqrt{(36-9x^2)}$
(3) Hyperbola	$\frac{x^2}{4}-\frac{y^2}{9}=1$	$y=\pm\frac{1}{2}\sqrt{(9x^2-36)}$
(4) Rectangular hyperbola	$xy=9$	$y=\frac{9}{x}$ $(x\neq 0)$

Note: The numbers 4 and 9 in the above equations may be replaced by any positive numbers a^2 and b^2 respectively.

Answers

EXERCISE 1

2. $\{x \mid -1\leq x\leq 1\}$ **3.** $\{x \mid -1\leq x\leq 2\}$

4. (i) $\{x \mid -1\leq x<1\}$ (ii) open

5. (i) $\{x \mid 1<x<2\}$ (ii) no values (iii) $\{2\}$

6. (i) $\{x \mid x>2\}$ (ii) all values of x
(iii) $\{x \mid x<2, \text{ but } x\neq -1\}$ (iv) $\{x \mid -3<x\leq 2\}$
(v) all values of x

7. (i) $x<90$; $x\leq 40$, $x\leq 10$; (ii) 1 hr., $\frac{1}{4}$ hr.

8. $-10\leq x\leq 10$

9. (i) $\{x \mid 2<x\leq 3\}$ (ii) $\{1\}$ (iii) $\emptyset$

10. (i) $\{x \mid -1<x\}$ (ii) $\{x \mid x>-1, x\neq 2\}$
(iii) $\{x \mid x>-2\}$ (iv) $\{x \mid -1\leq x<2\}$

11. (iii) **12.** (i)

EXERCISE 2

2. (i) $x>4$ (ii) $x<1$ (iii) $x>-6$ (iv) $x<4$ (v) $x>5$
(vi) $x<3$ (vii) $x<-10$ (viii) $x<-14$ (ix) $x<\frac{5}{2}$
(x) $x>-\frac{3}{2}$ (xi) $x>-\frac{7}{3}$ (xii) $x<\frac{6}{5}$

3. (1) $x>-5$ (ii) $x<4$ (iii) $x<-5$ (iv) $x<0$
(v) $x<-2$ (vi) $x<5$ (vii) $x<-18$

4. $x = 2$ **5.** (i) Yes (ii) No **6.** No
7. (i) No (ii) Yes **8.** $x<1$ or $x>2$
9. $-2<x<-1$ **10.** (i) $p<1$ (ii) $p>1$

EXERCISE 3

1. No restriction on x, $y>\frac{24}{13} = 1\cdot85$
2. $x = 2, y = 3$
3. $x>-\frac{8}{3}, y>\frac{2}{3}$
4. $x = -1, y = 2$
5. $y<-\frac{16}{13}$
6. (0A, 8B); (3A, 6B); (6A, 4B); (9A, 2B).
8. $\{x, y \mid 4x+y>0 \text{ and } 3x-2y<-3\}$
or $\{x, y \mid 4x+y>0\} \cap \{x, y \mid 3x-2y<-3\}$.

EXERCISE 4

1. $x+y<\frac{59}{11}$
2. $2y+5x<22\cdot4$
3. 1000, A 0, B
4. 3A, 9B
5. $2\frac{1}{2}$ hours on each.

EXERCISE 5

1. (i) 3 (ii) $\frac{2}{5}$ (iii) $\frac{4}{3}$
2. (i) $\frac{5}{2}$ (ii) $\frac{1}{4}$ (iii) 1
3. (i) $\frac{3}{5}$ (ii) $\frac{1}{2}$ (iii) $\frac{3}{7}$
6. -2
7. (i) $-\frac{1}{5}$ (ii) $-\frac{2}{3}$ (iii) $\frac{3}{4}$ (iv) $-\frac{3}{5}$

9. Gradient is price of sugar per lb.
11. 2π in/min.
12. 20 min after the 5-in depth.
13. 11.20 a.m.
14. 9·64 mile/h, 7·71 mile/h, 2h 36 min, after starting.
15. 24 mile/h.
16. $\frac{2}{3}$

EXERCISE 6

1. $x = 1, y = 1$
2. $x = 3, y = 6$
3. $\{(x, y) \mid 3x+5y = 8\} \cap \{(x, y) \mid x+4y = 5\}$

EXERCISE 7

4. (i) $\pm2\cdot24$ (ii) $1\cdot62$ or $-\cdot62$ (iii) $1\cdot69$ or $-1\cdot19$
(iv) $\pm2\cdot65$ (v) $2\cdot38$ or $-3\cdot38$ (vi) $3\cdot70$ or $-2\cdot70$
5. (i) 2·5 s (ii) 25 ft (iii) 1·12 s
6. $2\cdot19, -3\cdot19$; $x^2+x-7 = 0$
7. (i) $1, -1\cdot67$ (ii) $1\cdot52, -2\cdot19$ (iii) $-2\cdot28<x<1\cdot61$
(iv) $-1\cdot33$ (v) $0\cdot43, -0\cdot76$.
8. $1\cdot26$, $x>1\cdot26$
9. (i) $1\cdot30, -2\cdot30$ (ii) $2\cdot63, 0\cdot37$ (iii) $2\cdot54, -3\cdot54$
(iv) $\pm8\cdot37$ (v) $2, -4$.
10. $0\cdot27<x<3\cdot73$
11. 0·79
12. $0<x<1$ and $x<-1$ (i) No (ii) No
13. (i) $x+z = 100$ (ii) $x(100-x)$yd^2
(iii) 2,500 yd^2 (iv) square.